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Heisenberg-Inspired Quantum Automata

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- Quantum Computation is based on Hilbert spaces.
- We consider only finite dim. Hilbert spaces, i.e., vector spaces over $\mathbb{C}$.
- Column vectors are denoted by $|v\rangle$. Row vectors are denoted by $\langle v|$.
- States are unitary vectors $|v\rangle$.
- The systems evolve with linear unitary transformations U.

Example 1

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|v'\rangle = U|v\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle.$$

Measurement

- Measurements allow to extract information from quantum states.
- The extracted information is always classic (i.e., bits).
- Measurement operations are done through Hermitian operators.
- Before measuring, we can only compute the probability of seeing some outcome.

Example 2

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle \qquad P = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Probability of measuring 0 is $||P|v\rangle||^2 = |\alpha|^2$.

Measure-Once QFA

Definition 3 (Measure-Once QFA [4])

A QFA is a 5-tuple $M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where:

- Q is the finite canonical basis of $\mathbb{C}^d$ for some $d \in \mathbb{N}$;
- Σ is a finite alphabet;
- $\mathcal{U} = \{U_\sigma\}_{\sigma \in \Sigma}$ is a finite set of unitaries of dimension $\mathbb{C}^d \times \mathbb{C}^d$;
- $|\psi\rangle \in \mathbb{C}^d$ is the initial superposition of M;
- $F \subseteq Q$ is the set of final states. We define $P_F = \sum_{|q\rangle \in F} |q\rangle\langle q|$.

Example 4

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where $Q = \{|0\rangle, |1\rangle\}$, $\Sigma = \{a, b\}$, $\mathcal{U} = \{R(\theta), R(-\theta)\}$, $|\psi\rangle = |0\rangle$, and $F = \{|1\rangle\}$ is an example of MO-QFA.

- The unitary applied to the initial state for input $x \in \Sigma^*$ is

$$U_x = U_{x_n} U_{x_{n-1}} \cdots U_{x_1}$$

- The probability of a MO-QFA S to accept a string x is:

$$p_S(\mathbf{x}) = \|P_F U_{\mathbf{x}} |\psi\rangle\|^2 = \langle\psi| U_{\mathbf{x}}^\dagger P_F^\dagger P_F U_{\mathbf{x}} |\psi\rangle$$

Definition 5 (Cut-point QFA)

A language $L \subseteq \Sigma^*$ is *accepted by a QFA* S *with cut-point* λ if and only if $L = \{\mathbf{x} \in \Sigma^* \mid p_S(\mathbf{x}) > \lambda\}$.
A language $L \subseteq \Sigma^*$ is said to be *accepted by a QFA with cut-point* if and only if there exist a QFA S and $\lambda \geq 0$ such that $L \subseteq \Sigma^*$ is accepted by S with *cut-point* λ.

Example 6

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where $Q = \{|0\rangle, |1\rangle\}$, $\Sigma = \{a, b\}$, $\mathcal{U} = \{R(\theta), R(-\theta)\}$, $|\psi\rangle = |0\rangle$, and $F = \{|1\rangle\}$. M accepts the language $L = \{x \in \Sigma^* \ : \ |x|_a \neq |x|_b\}$ with cut-point 0.

Expressive power

Theorem 7 ([3])

Let L be a language accepted by a QFA S with cut-point λ. There exists a Probabilistic Finite Automaton that accepts L with cut-point λ'.

Using a pumping lemma from [2], QFAs enjoy the following property:

Corollary 8

QFAs can accept only languages that are either empty or infinite.

A perspective switch

Quantum Mechanics can be described through two pictures:

- Shrödinger picture, in which state evolves and observables are fixed.
- Heisenberg picture, where the state is fixed and observables change.

- An Heisenberg Quantum Finite Automata (HQFA) is defined as MO-QFA.
- The main difference is in its semantics.
- P_F is the current observable, σ the input symbol. $P'_F = U_\sigma^\dagger P_F U_\sigma$.
- The probability of a HQFA $\mathcal{H}$ to accept a string $x \in \Sigma^*$ is:

$$\rho_H(\mathbf{x}) = \|U^\dagger_{\overleftarrow{x}} P_F U_{\overleftarrow{x}} |\psi\rangle\|^2 = \langle\psi|U^\dagger_{\overleftarrow{x}} P_F^\dagger P_F U_{\overleftarrow{x}}|\psi\rangle$$

Example 9

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where $Q = \{|0\rangle, |1\rangle\}$, $\Sigma = \{a, b\}$, $\mathcal{U} = \{X, H\}$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $F = \{|0\rangle\}$ is an example of HQFA.

Expressive power

From the definitions of HQFA and MO-QFA, it is natural to notice that the relation between the two models is closely related to the notion of $\overleftarrow{L}$.

Theorem 10 (Mirror language)

Let $L \subseteq \Sigma^*$. L is accepted by a QFA with cut-point λ if and only if $\overleftarrow{L}$ is accepted by an HQFA with cut-point λ.

We now prove that QFAs are closed under the reverse operation.

Theorem 11 (Mirror Closure of QFAs)

Let $L \subseteq \Sigma^*$. L is accepted by a QFA with cut-point if and only if $\overleftarrow{L}$ is accepted a QFA with cut-point.

Final Results

Corollary 12 (Equivalence between QFAs and HQFAs)

Let $L \subseteq \Sigma^*$. L is accepted by a QFA with cut-point if and only if L is accepted by an HQFA with cut-point.

- Together with this result, in our proposal we also investigated some other models of Quantum Automata.
- In particular, we improved the current knowledge about Multi-letter Quantum Finite Automata [1].
- We also proposed a more general model that combines the ideas of Heisenberg and Multi-letter automata.

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