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Heisenberg in Quantum Automata

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Our Aim

From the Classical case...

- Automata are simple, yet interesting, Computation Models
- NFA/DFA characterize the languages decidable in Constant Space

...to the Quantum one

- (Dis)Advantages of Quantum Computation are still not precisely identified
- Studying the Expressive Power of Quantum Automata could provide important insights on the topic

Results

- We consider the simplest class of Measure-Once QFAs
- We analyse their expressive power under "small" changes:
  - We switch from Schrödinger's to Heisenberg's view of Quantum Mechanics
  - We enrich Measure-Once QFAs with both Bounded and Unbounded Memory on the prefixes
- We prove that Heisenberg QFAs have the same expressive power of standard Measure-Once QFAs
- We observe that QFAs with Unbounded Memory are not necessarily more expressive than Bounded Memory ones

Plan of the Talk

- Quantum Computation and Measure-Once QFAs
- Heisenberg Quantum Automata:
  - Closure w.r.t. Mirror Languages
  - Expressive Equivalence Theorem
- Bounded Memory Quantum Automata:
  - Pumping Lemma
  - Hierarchy Property
- Simplest Unbounded Memory Quantum Automata:
  - Negative result on the Expressive Power

Quantum States and Evolution

- Quantum Computation is based on Hilbert spaces
- Finite dimension Hilbert spaces are vector spaces over $\mathbb{C}$

- Column vectors are $|v\rangle$ and Row vectors are $\langle v|$
- States are unitary vectors $|v\rangle$

- Unitary transformations U rule the evolutions:
  - linear transformations
  - preserve length and angles

Example of Evolution

Example 1

$$|v\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \qquad U = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$|v'\rangle = U|v\rangle = \frac{-i}{\sqrt{2}}|0\rangle + \frac{-1}{\sqrt{2}}|1\rangle$$

2×2 Unitary matrices are rotations on the Bloch Sphere.

- Measurements/Observables allow to extract information from quantum states
- The extracted information is always classic (i.e., bits)
- Measurement operations are done through Hermitian operators
- Before measuring, we can only compute the probabilities of the outcomes

Example 2

$$|v\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|v'\rangle = U|v\rangle = \frac{-i}{\sqrt{2}}|0\rangle + \frac{-1}{\sqrt{2}}|1\rangle$$

Example 3

$$P = |0\rangle\langle 0| = (1 \;\; 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Probability of the outcome 0 after measuring $|v'\rangle$

$$||P|v'\rangle||^2 = \left|\frac{-i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Definition 4 (Measure-Once QFA [2])

A MO-QFA is a 5-tuple $M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where:

- Q is the finite canonical basis of $\mathbb{C}^d$ for some $d \in \mathbb{N}$
- Σ is a finite alphabet
- $\mathcal{U} = \{U_\sigma\}_{\sigma \in \Sigma}$ is a finite set of unitaries of dimension $\mathbb{C}^d \times \mathbb{C}^d$
- $|\psi\rangle \in \mathbb{C}^d$ is the initial superposition of M
- $F \subseteq Q$ is the set of final states. We define $P_F = \sum_{|q\rangle \in F} |q\rangle\langle q|$

Example of a MO-QFA

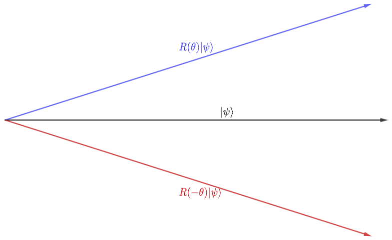
Example 5

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where

$Q = \{|0\rangle, |1\rangle\}$ $\qquad$ $\Sigma = \{a, b\}$

$\mathcal{U} = \{U_a = R(\theta), U_b = R(-\theta)\}$ $\qquad$ $|\psi\rangle = |0\rangle$

$F = \{|1\rangle\}$



- The unitary applied to the initial state for input $\mathbf{x} = x_1 \ldots x_{n-1} x_n \in \Sigma^*$ is:

$$U_{\mathbf{x}} = U_{x_n} U_{x_{n-1}} \cdots U_{x_1}$$

- The probability of a MO-QFA S to accept a string $\mathbf{x}$ is:

$$p_S(\mathbf{x}) = \|P_F U_{\mathbf{x}} |\psi\rangle\|^2 = \langle\psi| U_{\mathbf{x}}^{\dagger} P_F^{\dagger} P_F U_{\mathbf{x}} |\psi\rangle$$

Acceptance Conditions

Definition 6 (Cut-point QFA)

A language $L \subseteq \Sigma^*$ is accepted by a QFA S with cut-point λ if and only if $L = \{\mathbf{x} \in \Sigma^* \mid p_S(\mathbf{x}) > \lambda\}$

A language $L \subseteq \Sigma^*$ is said to be accepted by a QFA with cut-point if and only if there exist a QFA S and $\lambda \geq 0$ such that $L \subseteq \Sigma^*$ is accepted by S with cut-point λ

Example 7

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where

$Q = \{|0\rangle, |1\rangle\}$ $\Sigma = \{a, b\}$

$\mathcal{U} = \{R(\theta), R(-\theta)\}$ $|\psi\rangle = |0\rangle$

$F = \{|1\rangle\}$

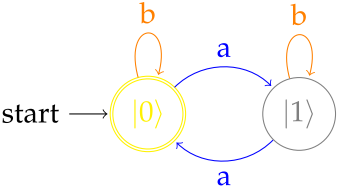
M accepts the language

$$L = \{x \in \Sigma^* \ : \ |x|_a \neq |x|_b\}$$

with cut-point 0

$L = \{x \in \{a, b\} \mid x$ has an even number of as$\}$

Classical case



Quantum case

$Q = (\{|0\rangle, |1\rangle\}, \{a, b\}, \{U_a, U_b\}, |0\rangle, \{|0\rangle\})$ where

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $U_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $U_b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $P_F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Quantum Mechanics can be described through two pictures:

- Schrödinger picture, in which the state evolves and observables are fixed
- Heisenberg picture, where the state is fixed and observables change

- An Heisenberg Quantum Finite Automaton (HQFA) is defined as MO-QFA
- The semantics is different
- P_F is the current observable, σ the input symbol:

$$P'_F = U^\dagger_\sigma P_F U_\sigma$$

- The probability of a HQFA $\mathcal{H}$ to accept a string $\mathbf{x} \in \Sigma^*$ is:

$$\rho_H(\mathbf{x}) = \| U^\dagger_{\overleftarrow{x}} P_F U_{\overleftarrow{x}} |\psi\rangle \|^2 = \langle \psi | U^\dagger_{\overleftarrow{x}} P^\dagger_F P_F U_{\overleftarrow{x}} |\psi\rangle$$

Example 8

$M = (Q, \Sigma, \mathcal{U}, |\psi\rangle, F)$ where

$Q = \{|0\rangle, |1\rangle\}$ $\Sigma = \{a, b\}$

$\mathcal{U} = \{U_a = X, U_b = H\}$ $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$F = \{|0\rangle\}$

If M is a MO-QFA:

$$p_M(ab) = \| |0\rangle\langle 0|U_b U_a|+\rangle\|^2 = \| |0\rangle\langle 0|U_b|+\rangle\|^2 = \| |0\rangle\langle 0||0\rangle\|^2 = 1$$

If M is a HQFA:

$$\rho_M(ab) = \|U_b^\dagger U_a^\dagger |0\rangle\langle 0|U_a U_b|+\rangle\|^2 = \|U_b^\dagger |1\rangle\langle 1|U_b|+\rangle\|^2 = 0$$

$$\rho_M(ba) = \|U_a^\dagger U_b^\dagger |0\rangle\langle 0|U_b U_a|+\rangle\|^2 = \|U_a^\dagger |+\rangle\langle +|U_a|+\rangle\|^2 = 1$$

HQFAs and MO-QFAs are related through mirror operation $\overleftarrow{L}$

Theorem 9 (Mirror language)

L is accepted by a MO-QFA with cut-point λ if and only if $\overleftarrow{L}$ is accepted by an HQFA with cut-point λ

MO-QFAs are closed under the mirror operation

Theorem 10 (Mirror Closure of MO-QFAs)

L is accepted by a MO-QFA with cut-point if and only if $\overleftarrow{L}$ is accepted by a MO-QFA with cut-point.

Corollary 11 (Equivalence between MO-QFAs and HQFAs)

L is accepted by a MO-QFA with cut-point if and only if L is accepted by an HQFA with cut-point

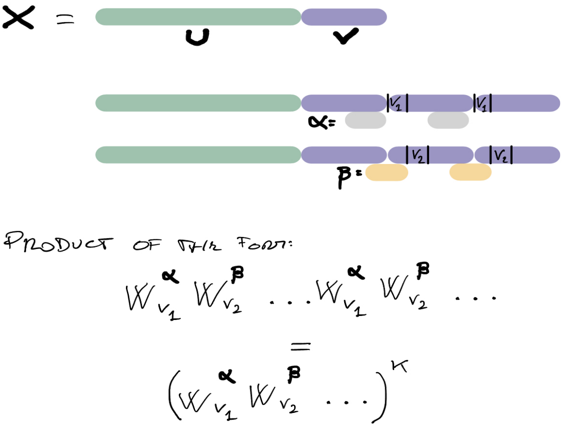
- The results so far do not increase the expressive power of MO-QFAs
- We investigated a model were the notion of memory is introduced
- We considered h-MQFA (equivalent to [1])
- Unitaries now depend on suffix prefixes of length at most h

STATE BEFORE x_j : $|\psi\rangle$

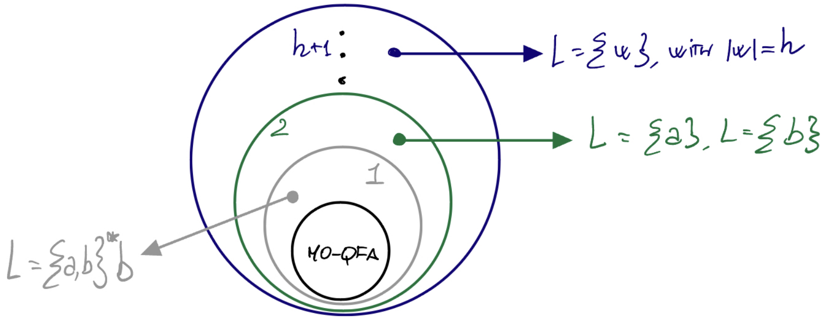
STATE AFTER x_j : $|\psi\rangle = W^{Y}_{x_j} |\psi\rangle$

Theorem 12 (Pumping Lemma for h-MQFAs)

Let $L \subseteq \Sigma^*$ be the language accepted by an h-MQFA
$\forall \mathbf{uv} \in L$ with $|\mathbf{v}| \geq h$ there exists $k \in \mathbb{N}^+$ such that $\mathbf{uvv}^k \in L$



Hierarchy Property



Unbounded Memory

- Even with finite memory, some regular languages could not be accepted.
- Therefore we tried to give an Unbounded Memory
- We called the resulting automaton model UMQFA
- The definition is the same as h-MQFAs, but the semantic is different
- Unitaries now depend on the prefixes of the input

Computation in UMQFAs

STATE BEFORE x_j : $|\varphi\rangle$

STATE AFTER x_j : $|\varphi'\rangle = U_{x_j}^{y} |\varphi\rangle$

STATE AFTER x_{j+1} : $|\varphi''\rangle = U_{x_{j+1}}^{v} |\varphi'\rangle$

UMQFAs Limitations

By studying the sequence $\{V^{\frac{k(k+1)}{2}}\}_{k>1}$, where V is a square complex matrix, we obtained the following result about UMQFA expressiveness

Theorem 13

Let $\Sigma = \{a\}$ and $L = \{\epsilon, a\}$
There is a 2-MQFA that accepts L with cut-point and there is no UMQFA accepting L with cut-point

Conclusions

- We played with some variants of Measure-Once QFAs
- We fought against the limitations imposed by Unitaries
- We proved some Closure and Equivalence results on Measure-Once QFAs
- We characterized Measure-Once QFAs with Bounded Memory on the prefixes
- The Unbounded Memory case needs further investigations

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